TOWARDS A MATHEMATICAL MODEL FOR SEMANTICS.

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In this paper an attempt will be made to offer a non-technical exposition of the principles underlying the author's recent work in constructing a mathematical model for language in which full use of semantic concepts is made. In order to simplify the presentation attention will be restricted primarily to the level of the simple sentence and the word will be assumed to be the minimal, free, meaningful unit.

As a formal model of a real phenomenon, the present structure does not pretend to to offer a complete or perfect description of the situation that it seeks to model, but rather examines those aspects accessible to analysis in the way that mathematical physics analyses certain aspects of the physical universe. If such a model is to be worthwhile it must have powers of definition and explanation of observed phenomena and should allow deductions of new facts to be made. Lacking detail, it should seek power in generality, that is, in achieving the widest possible range of validity.

This present work should be considered as representing an early stage of development and hence as being liable to much future modification. The approach used is not unlike that of Abraham and Kiefer (1966) but the mathematical structure is deeper and, in the author's opinion, is more flexible and capable of development.

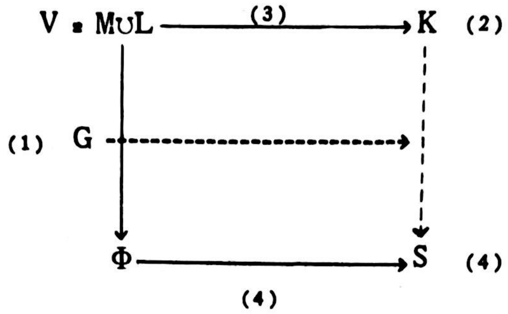
We shall consider a language from the following point of view. Every language will have a vocabulary (V) each of whose members, called words, may be either grammatical markers (M) or a designatum for an external concept. This latter set of words we shall call the lexicon (L). In general, some words may play both rôles simultaneously.

By means of the operation of concatenation, strings of words may be formed and a proper subset (Φ) of the set of all possible strings will be called the set of sentences or marked strings of the language. In writing a grammar for a given language we wish to specify precisely which strings belong to Φ and to give to these strings a structural description (Chomsky 1965a, 1965b).

A primary feature of the present model is an attempt to reduce these structural descriptions to a small number of basic types each of which fulfills a particular semantic function. This is similar to Chomsky's structure of "kernel sentences" from which other sentences may be derived by means of transformations. In our case, transformations will be used with a semantic rôle; it will be convenient, for example, to consider negation as a transformation whether or not the grammatical complexity of the negative sentence would be considered, by the Transformationalists, to merit the use of a transformation rule.

To summarize the model at this stage we note the following features: (1) A grammar (over V) generating structural descriptions which are partitioned into certain basic sentence types.
(2) A set of concepts (K) corresponding to the designata.
(3) A correspondence between the lexicon and the set of concepts.

(4) A construction on the set of concepts corresponding to each of the various sentence types (S).



Clearly, little progress can be obtained from this wide viewpoint without a consideration of the form of the structure of Φ, K and S. We shall firstly consider the space of concepts K.

In psychological terms, concepts are developed throughout life, and especially in the early part of life, by processes of linkage and generalization on a few basic concepts or ideas. Such processes are accomplished mainly, but not exclusively, by linguistic means and they can normally be expressed linguistically or in a quasi-linguistic or iconic form as in the arts. By linkage we mean, roughly, the process of forming the union or join of concepts, and by generalization, the process of taking the concepts common to several situations.

With this situation in mind, we posit K as a set consisting of a finite set of basic points (basic concepts) together with all the finite subsets generated by these points and then all the finite subsets generatable from these points *et seq.* The resulting infinite set of objects we take as the points of the space K and we must next consider structures over this set of points.

One structure on K becomes immediately obvious, viz., the partial ordering of the points under the relation of set inclusion (see appendix). This structure is intended to model logical and phenomenological structures on concepts such as in the chain, "every *Male* is *Human* is *Mammal* is *Animal* is......" where the word "is" plays the role of the inclusion operator. Such a relation defines a directed graph (see appendix) over K and this in turn describes a topology (structure) on K in which concepts are linked together by their common properties. Further structures will be imposed on K later.

We shall approach the problem of the structure on Φ by considering the well-known problem which faces a linguist when investigating an unknown language. His data is a corpus of material in the form, effectively, of linear strings of symbols which, upon examination, prove to be structured. Using techniques such as those expounded by Harris (1947) he will compare strings (utterances) noting where one symbol may be replaced by another without producing a non-marked string.

It is easy to see that mutual substitutability of words in marked strings defines an equivalence relation (see appendix) which divides the vocabulary into mutually disjoint word classes. In analytical languages such as English or Chinese these classes will correspond closely to the traditional concept of "parts of speech" but in inflected languages a more complex relation, which takes account of paradigms, must be defined in order to accomplish the same result. Again, in agglutinative languages such an analysis may partition morphemes rather than words but, with appropriate modifications, such a partitioning can be achieved and the resulting classes often have characterising semantic properties. If such a set of word classes can be obtained, we can derive a form of structural description of our marked strings, viz., sequences of classes corresponding consecutively to each word of the original string. It is amongst these sequences that we must search for our basic sentence types. Analysis of such structures, called B-structures, has been extensively considered by Revzin (1962) and others, and a mathematical analysis has been published by Marcus (1967).

We may note, at this point, that the mathematical techniques by which these B-structures are established and the techniques by which these B-structures may be analysed into basic types are formally identical, a feature which is of help in the development of the mathematical structure of the model.

Although this is a practical technique which may be used to analyze real languages without the use of semantic notions, no algorithm has ever been offered for formalising the process. Now formal, or mathematical languages fall within the scope of our general definition of language and it is well known that many of these have an undecidable word problem; that is, there is no effective process available or constructable which will determine, for any arbitrary string of words, whether or not that string is a marked string. Thus we have no *a priori* certainty that a purely syntactic analysis of a language (in this sense) will yield such a partition into word classes. That all known, colloquial languages can be so partitioned is symptomatic of the rather obvious fact that language has non-syntactic properties. It is also surely true that no field worker can but take account of the semantic nature of the language he is studying if he is to make any analytical progress.

Assuming that an analysis of the marked strings into B-structures has been achieved, at least in part, the resulting system may then be further analysed by type-theory techniques such as those introduced by Bar Hillel (1953), Lambeck (1961), and others. In this technique, the marked string may be given a type 's' and the class of nouns (say) a type 'n'. Other word classes may then be assigned types according to the contexts in which they occur. This results in an algebraic algorithm which determines which B-structures yield marked strings - provided the analysis is effectively possible. Where such an analysis has succeeded it leads in a natural way to a definition of a Phrase Structure generative grammar in the sense of Chomsky. Even as the process of syntactic analysis is more likely to achieve success if it is restricted to certain classes of relatively simple sentences, so a convenient grammar may be written, following Chomsky, by using Phrase Structure generators for a limited set of kernel sentences (together with the help of morphophonetic transforms), and then generating the remaining sentences by transforms both of a semantic and of a purely syntactic form.

Let us now consider the relationship between the two systems that we have so far considered, viz., the semantic and the syntactic.

To each item of the lexicon there corresponds a set of 'readings' in the form of a set of concepts, that is points of K. These readings are defined by a 'dictionary' and each separate dictionary reading is considered as a separate point of K. If true synonyms exist, then we shall also have the situation that to some points of K there correspond several lexical items.

Since our vocabulary is assumed to be partitioned, this correspondence will induce a partitioning on K. Presumably, in a very orderly language where each word class has a distinct semantic rôle, this partitioning would coincide with certain of the structures already existing on K; for example, where nouns always refer to the logical category of objects and verbs to that of actions, say. In practice this ideal situation will rarely obtain.

It is very significant that any language for which a syntactic analysis is possible will have a semantic model of the type so far outlined, provided of course that a dictionary exists. Conversely, the detailed mathematical structure envisaged for this model would guarantee the existence of a successful syntactic analysis. In other words a language will have a decidable word problem if and only if it has a semantic model. Apart from the significance of this result to mathematics it would also justify the use of an appeal to semantic notions, in principle at least, even to the most strict of formal structuralists.

By analogy with syntactic analysis we may reduce the number of basic partitionings in K. Consider, for example, the case of a transitive verb. In essence, a given verb is completely determined by two sets of words (or groups of words with a specified structure) viz., the set of its possible subjects and the set of its possible objects. Thus if, say, we denote the set of nominal forms by N, then a given verb is identical with a specified subset of the Cartesian product (see appendix) NxN. In practise it may be necessary to include an additional semantic marker to distinguish closely related verbal concepts of this type. In the same way, an intransitive verb would consist of a subset of N together with a specifier and it may well be that modifying classes can be similarly treated.

Traditionally, such a class of nominal forms was defined by indicating a property or properties that all the members of that class must satisfy. In the present model such classes are automatically defined, in a natural way, within the aforementioned topology on K and moreover such a reduction, wherever possible, gives an automatic indication as to which strings of words make semantic, or at least logical sense.

A further structure on K now appears. In addition to logical and phenomenological relations between concepts, there are also relations which are established by experience and by sociological conditioning, relationships which vary from one individual to another and from one culture to another. Such relationships can be linguistically established by adequate repetition of pairs of words, or of chains of pairs of words in context; a process reminiscent of that by which the brain itself becomes conditioned. Such a relation, being reflexive, defines an undirected graph over K (see appendix), a structure which could be the object of a psychological study such as that by Osgood et al. (1957). Concept structure is therefore defined as being dependent on two relational structures and may, in any individual or society, contain inconsistencies. It should be noted that if we are able to make reductions of the type described above in respect of the transitive verb etc., the appropriate relations of this latter type are automatically induced on the given verbal concept.

We now come to a consideration of the structure that must model the semantic status of our complete sentence. Any sentence will consist of a string, some of whose symbols (such as order or inflexion or particles) serve simply to mark grammatical function.

Other symbols, belonging to the lexicon, correspond to concepts and so each sentence structure (B-structure) when provided with a set of arguments from the lexicon, posits a certain linkage (semantic function) between the corresponding concepts.

To model these linkages we construct the space S in the following manner. Let $W_1, W_2, \ldots, W_k$, be the word classes of the given language. To each class W_i, $(i = 1,2,\ldots, k)$ we have assigned a directed graph L_i (the partial ordering described as the first structure on K) and also an undirected graph C_i (the second graph structure described on K). We define next the Cartesian product of graphs (see appendix) and it is upon this that our final structure will be based. Since both the graphs L_i and C_i have the same vertices, we can consider them as a single graph G_i say. We now take a product of the graphs G_i corresponding to each B-structure (sentence structure); e.g., if the B-structure '$W_3W_6W_4W_6$' occurs, then we must take a product '$G_3 x G_6 x G_4 x G_6$'. In a detailed construction we would, in fact, begin this process at phrase level and work upwards, there being no technical reason why the process cannot be continued up to discourse level. Further, where we have made use of semantic transforms in the syntactic part of our analysis, we correspond to each such transform T a concept C_T. Then if a given B-structure corresponds to a graph $\bar{G}$ the transform under T of that sentence structure will correspond to the graph $\{C_T\} x \bar{G}$. Thus all sentences, half sentences, clauses and phrases, etc., will correspond to points in S and it is claimed that the very rich structure of this space already models many semantic properties of language not explicitly built into it. The justification of such a claim is necessarily mathematical but, in conclusion, this paper will attempt to give a brief description of the manner in which various semantic phenomena appear in this model.

Perhaps, in passing, it is worth noting that any conceptual situation which is capable of linguistic expression will be representable within this model.

(1) *Ambiguity and its resolution.* Where a word, occurring in a sentence, has several dictionary readings, that sentence will generate correspondingly many points in the space S. In the case of a semantically inadmissible reading, the relevant L-graphs of subsentence structure will not be connected (see appendix). Note that this implies that the problem of grammatically correct but anomalous sentences can be dealt with at the semantic level. In the case of admissible readings, the graphs will be connected but, in general, the paths leading to the different readings will be of differing lengths, the shortest path length giving the most probable reading. In the case of discourse analysis the most probably correct reading will be that which has the shortest C-graph path to the points corresponding to other sentences in the discourse. In the case of allusive and literary language, points may be disconnected on the C-graph but there is some evidence to suggest that an interpretation can be made by searching for paths of minimal length on the L-graph considered as being undirected.

(2) *Semantic information.* A similar consideration will give us a measure of semantic information (Bar Hillel 1952). A well known fact, the iteration of which will give little information, will correspond to a conceptual linkage (a short path on the C-graph structure), whereas a surprising fact will generate a longer path. Thus path length can serve as a direct measure of semantic information.

(3) *Translation.* A successful translation between languages is one where 'meaning' is preserved as faithfully as possible. It follows that if we can construct a seman-

tic (S) space for each of our two languages, then the problem reduces to one of mapping the one space onto the other and so the question of translatability becomes one of the degree of isomorphism of these two spaces. By construction the basic concept space K will be identical in both cases.

However, the two different languages, developing under differing cultural situations, will have chosen, in general, to append words to different subsets of K. Thus we can consider which words of the one language are replaceable by words of the second, and also which words correspond to concepts which, in the second language will have to be introduced by definition or circumlocution. Secondly, we can compare the grammars, i.e., compare the graph products needed in each case and consider which give the best mutual approximations in various cases. Thirdly, we may consider the C-structures to see what, if any, conceptual barriers there may be to a successful translation. It is immediately clear that the chances of achieving complete isomorphism are small.

(4) *Alternative grammatical theories.* Alternative descriptions of a single language can be considered as a comparison of two 'different' languages where the vocabularies, dictionaries, C-graphs and L-graphs happen to coincide. To prove or disprove the equivalence of the two descriptions then reduces to the problem of proving the isomorphism of the product structures corresponding to the different sentence structures. Consider, by way of example, a grammar which introduces a simple English negation by means of a phrase structure apparatus and an alternative grammar which uses a transformation for the same purpose. In the one case we may consider the word 'not' as an adverb and in the second case as an isolated concept. If then Av is the set of adverbs, V the set of verbs and 'neg' denotes the concept of negation we have to prove that the two sets $(Av - \{neg\}) x \{neg\} x V$ and $\{neg\} x (Av \times V)$ are isomorphic. This is, in fact, a particular case of a more general theorem on graphs which could obviate much polemics on the part of rival theorists.

In conclusion the author would like to apologize for the vagueness of certain sections of this paper resulting from a non-technical discussion of an essentially mathematical treatment.

Appendix of mathematical terms.

A *set* is a collection of objects. It is itself a single object. A set may be denoted by displaying names of its members within braces, thus {the, a, an}, or by giving a property or properties satisfied by all the members of the set and only by those objects. e.g. 'the set of singular articles in English'. Note that $\{C_T\}$ denotes the set whose only member is the object C_T.

The *union* of two sets A and B, written $A \cup B$, is that set consisting of the totality of the objects in A and those in B; hence an object is in $A \cup B$ if and only if it is either in A or in B or in both A and in B.

The *intersection* of two sets A and B, written $A \cap B$, is that set consisting of those objects common to A and to B.

The *empty set* is that set which contains no members.

Two sets are said to be *mutually disjoint* if their intersection is the empty set.

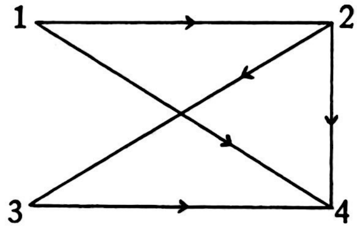
A-B is the set consisting of all those objects in A which are not in B. If every member of a set A is also a member of the set B we write $A \subset B$. We say that A is a subset of B. For two sets A and B, A = B if and only if $A \subset B$ and $B \subset A$.

The Cartesian product AxB of A with B is the set of all pairs (a,b) of objects, where a belongs to A and b belongs to B; e.g. if A = {1,2} and B = {a,b}, then AxB = {(1,a), (1,b), (2,a), (2,b)}.

A *relation* on a set A is a subset of AxA. If R is such a relation, $R \subset AxA$, and if (a,b) belongs to R we may write aRb. Note that this does not imply bRa. Such a relation is symmetric if, for all a in A, aRa holds; it is *reflexive* if bRa holds whenever aRb holds; it is *transitive* if, whenever both aRb and bRc hold, then aRc must hold, for a,b,c belonging to A. If all these three properties hold, then the relation is called an *equivalence* relation and it then has the important property of partitioning the set A into mutually disjoint sets called *equivalence classes*.

A *graph* G = {V,E}, consists of a set of points V called vertices and a set E of edges, where $E \subset VxV$. It may be represented by means of a diagram showing V as points and representing E by lines joining the appropriate pairs of vertices. Note that E is a relation on V and hence relational systems can be considered as graphs.

e.g. Let G = {{1,2,3,4}, {(1,2), (2,3), (2,4), (1,4), (3,4)}}. This may be represented in the following diagram.



If the relation E is not reflexive we say that the graph is *directed* and represent the edges by arrows as in the present diagram. If E is reflexive we say that the graph is undirected and can represent the edges simply by lines. A *path* on a graph is a sequence of adjoining edges, taken only in the correct direction in the case of a directed graph; e.g., in the above diagram there is a path from 1 to 2 to 3 to 4, but there are no paths from 4 to any other point.

A graph is *connected* if there is a path joining any two vertices.

The *Cartesian product* of two graphs is constructed as follows: if the two graphs are $G_1 = \{V_1,E_1\}$ and $G_2 = \{V_2,E_2\}$ and $G_1xG_2 = G$, then the vertices of G consist of the set V_1xV_2; if (a,b) and (c,d) are two vertices of G_1xG_2 then ((a,b), (c,d)) is an edge if and only if either a = c and (b,d) is an edge of G_2, or b = d and (a,c) is an edge of G_1. What this construction does, in effect, is to provide a copy of G_2 at each vertex of G_1 and a copy of G_1 at each vertex of G_2.

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